b5 Differential and Integral Equations - ht99 Vacation Work

Plane autonomous systems of ordinary differential equations: phase-plane and critical points, linear systems, elementary non-linear systems and linearisation. Bendixson-Dulac criterion.

1. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) && (1)\\ \frac{dy}{dt} &= g(x,y) && (2)\\ & && (3) \end{aligned}$$

   Explain the significance of the *trajectories* (the solutions of $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$). Explain the significance of the singular points[1] (x_s, y_s) where $f = g = 0$.

   Suppose (1)–(2) has a singular point at $(0,0)$, and f and g are analytic[2] near $(0,0)$. Then we can expand f and g in Taylor series about the origin. Do so, then linearise the resulting equations and write in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} \qquad (4)$$

   Since this is a linear equation, try solutions $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\mathbf{w}}e^{\lambda t}$ [*cf* ODEs for which we try $y(x) = ce^{mx}$]. Find the general solution of (4) and explain the significance of the eigenvalues and eigenvectors of A.

   Prove that $(0,0)$ is stable $\iff$ Tr $A < 0$, Det $A > 0$.

   Explain how these results generalize to *any* singular point.

2. Consider the Lotka-Volterra equations for the population of an island of rabbits ($r \geq 0$) and sheep ($s \geq 0$). After the population numbers are scaled suitably, the sheep population satisfies $\frac{ds}{dt} = s(2 - r - s)$.

   In each of the following cases for the rabbit equation, find the critical points in the (r,s) phase plane, classify them (as one of saddle, stable/unstable node/focus) and sketch a plausible phase portrait compatible with what you find. Make sure that you work out the eigenvectors (and that you know why you need to do this, and that you *use* them).

   In each case, state briefly the ecological implications of your findings. Do you see anything about the steady states on the axes that might at first seem odd? Explain why the trajectories here do not contradict common sense after all.

   (a) $\frac{dr}{dt} = r(3 - r - s)$
   (b) $\frac{dr}{dt} = r(3 - 2r - s)$
   (c) $\frac{dr}{dt} = r(3 - r - 2s)$
   (d) $\frac{dr}{dt} = r(3 - 2r - 2s)$

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[1] Also known as critical points or steady states.
[2] Please state what analytic means!

Fredholm alternative (proof for finite rank kernels only).

3. Consider the integral equation

$$y(x) = f(x) + \underbrace{\lambda \int_a^b K(x,t)y(t)dt}_{(*)} \tag{5}$$

Suppose that the kernel K is degenerate, *ie,* it can be written as a finite linear combination of products of a function of x and a function of t. Write the preceding statement in mathematical notation.

Use this to show that term $(*)$ in (5) can be written as a linear combination of known functions of x, with coefficients which are unknown constants X_j. Thence show that (5) may be written as a system of linear algebraic equations

$$(\mu I - A)\underline{\mathbf{X}} = \underline{\mathbf{b}} \tag{6}$$

where $\underline{\mathbf{X}} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$, $\mu = \lambda^{-1}$ and A and $\underline{\mathbf{b}}$ should be given. State conditions under which (6) has

(a) a unique solution

(b) no solutions

(c) many solutions

and explain why the solutions of (6) for $\underline{\mathbf{X}}$ are relevant to the solutions of (5) for $y(x)$.

Explain what we mean by the *eigenvalues* and *eigenfunctions* of (5). Explain how these affect the possible solutions of (5)[3].

4. In the previous question, we worked through a method for investigating solutions of the general non-homogeneous Fredholm equation (5). In this question, we will apply these methods to the particular problem

$$y(x) = f(x) + \lambda \int_{-\frac{\pi}{2}}^{\frac{pi}{2}} K(x,t)y(t)dt$$

where $K(x,y) = \sin x \sin y + \sin x \cos y + \cos x \cos y$ and f is continuous.

Show that the only characteristic value[4] of K is $2/\pi$, and find the corresponding space of characteristic functions[5]. If $\lambda = 2/\pi$, what condition(s) does f have to satisfy for a solution to exist?

Solve the equation when $\lambda \neq 2/\pi$.

[3] *ie,* state the Fredholm alternative.

[4] Another term for eigenvalue.

[5] 'Characteristic functions' is another term for 'eigenfunctions'. You're being asked to give the general form of an eigenfunction.

Hilbert-Schmidt theory for equations with symmetric kernels: orthonormal families, Bessel's inequality and eigenvalues, the expansion theorem (proof of the existence of an eigenvalue excluded).

5. Consider the homogeneous Fredholm equation

$$y(x) = \lambda \int_a^b K(x,t)y(t)dt \tag{7}$$

where K is real-valued, continuous and symmetric.

(a) Explain what we mean by the eigenvalues and eigenfunctions of K. It can be shown that (7) has at least one eigenvalue.

(b) Show that the eigenvalues are real, and that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(c) Show that every eigenvalue has at least one *real* eigenfunction.

(d) Show that each eigenvalue has *finite multiplicity*, *ie,* there can be only a *finite* linearly independent set of eigenfunctions corresponding to each eigenvalue.

(e) Show that (7) has a countable number[6] of distinct eigenvalues.

(f) Use the above results to explain how we can enumerate a set of real orthonormal eigenfunctions $y_1, y_2, \ldots$ corresponding to a set of real (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots$.

Clearly state the Hilbert-Schmidt Expansion Theorem for a uniformly convergent series expansion of the continuous function $f(x)$, where

$$f(x) = \int_a^b K(x,t)g(t)dt$$

stating clearly the conditions required on K and g, and making clear how the coefficients of the series expansion are found.

6. Let $f(x)$ be a continuous real-valued function of period 2π, and let a_n be its Fourier cosine coefficients for $n \geq 0$:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

Let $K(x,t) = f(x-t)$ for $x, t \in [0, 2\pi]$. Show that if $a_n \neq 0$ then $\cos nx$ and $\sin nx$ are eigenfunctions of K and state the corresponding eigenvalues. You may assume that these eigenvalues are the *only* eigenvalues, and that the corresponding $\cos nx$, $\sin nx$ span the space of all eigenfunctions.

Assuming the Hilbert-Schmidt expansion theorem[7], show that for any continuous $u(x)$ on $[0, 2\pi]$,

$$\int_0^{2\pi} f(x-t)u(t)dt = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx$$

where the series converge uniformly in x. Express A_n and B_n in terms of a_n and u.

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[6] Which may be finite.

[7] Which in an isolated question you should state, but you've done this in the previous question; if you use a different form here, however, please state it.

Iterative methods and Neumann series applied to Fredholm integral equations.

7. Consider the non-homogeneous Fredholm equation (5). Define the *iterated kernels* and explain their significance. State the solution of (5) in terms of the *Neumann series*, and sketch a proof that this is in fact the solution. Give a necessary and sufficient condition for uniform convergence of the Neumann series (the proof of this is not on your syllabus).

8. Consider (5) with $K(x,t)$ as in Question 4. Show that the iterated kernels are given by

$$K_n(x,y) = \left(\frac{\pi}{2}\right)^{n-1} (\sin x \sin y + n \sin x \cos y + \cos x \cos y)$$

Show directly (*ie,* not by quoting the 'neccessary and sufficient' condition above) that the Neumann series for y converges when $|\lambda| < 2/\pi$, and verify that the sum of the series agrees with the solution you have already found in Question 4.

Calculus of variations: Euler's equation as a necessary condition. Extension to a finite number of dependent functions and their derivatives, a finite number of independent variables, integral constraints and variations of boundary values. Applications to geometry, mechanics and quantum mechanics.

9. Use the Fundamental Lemma of the Calculus of Variations (which you should state) to show that a function $y(x)$ which minimises

$$I := \int_a^b F(x, y, y')\, dx$$

over all twice-continuously differentiable functions satisfying given boundary conditions, must also satisfy Euler's equation

$$\frac{d}{dx}(F_{y'}) = F_y$$

10. In geometrical optics, the light ray between two points A and B in three-dimensional space satisfies Fermat's principle; *ie,* it minimises

$$I := \int_A^B \mu\, ds$$

where μ is the refractive index (a positive function of position) and ds is the element of distance along the ray.

Suppose that μ depends only on the height z. Explain why the ray between A and B will lie in the vertical plane containing them.

Now suppose that $\mu = \sqrt{z}$, and suppose that A, B and the ray between them lie in the region $z > 0$. Use the calculus of variations to show that the ray satisfies $z^2 = (z - 2K)^2 + (x - \lambda)^2$, where λ and $K \geq 0$ are constants determined by A and B.

If $A = (-a, 0, c)$ and $B = (a, 0, c)$, explain why we must have $\lambda = 0$, and write down an equation that K must satisfy.

(a) If $a < c$ then the equation for K has two solutions. Which of them gives the correct minimum value for I, and why? Find this minimum value.

(b) If $a > c$ then the equation for K has no solution. What does this imply physically?

11. (a) Consider the problem of minimising

$$I := \int_a^b F(x, y, z, y', z')\, dx$$

Sketch a proof that minimisers $y(x)$ and $z(x)$ must satisfy the Euler equations, which you should state.

(b) Consider minimising

$$I := \int_A\int F(x, y, u, u_x, u_y)\, dx\, dy$$

over all twice-continuously-differentiable functions $u(x, y)$ satisfying $u(x, y) = f(x, y)$ on ∂A. Sketch a proof that a minimiser u must satisfy

$$\frac{\partial}{\partial x}\left[F_{u_x}\right] + \frac{\partial}{\partial y}\left[F_{u_y}\right] = F_u$$

stating conditions on A, ∂A, F and f for your proof to work, and pointing out where in the proof you use each condition.

(c) Consider minimising

$$I := \int_{t_0}^{t_1} F(x, y, z, \dot{x}, \dot{y}, \dot{z})\, dt$$

over curves $\underline{\mathbf{r}}(t) = (x(t), y(t), z(t))$ passing through $\underline{\mathbf{r}}(t_0)$ and $\underline{\mathbf{r}}(t_1)$, and lying completely in the surface $z = g(x, y)$.

Show that the problem reduces to minimising

$$J := \int_{t_0}^{t_1} H(x, y, \dot{x}, \dot{y})\, dt$$

over curves $x(t)$ and $y(t)$ passing through $(x(t_0), y(t_0))$ and $(x(t_1), y(t_1))$, where H is a function which you should state in terms of F.

12. A point moves in the plane from (x_0, y_0) at time t_0 to (x_1, y_1) at time t_1 in such a way that

$$I := \int_{t_0}^{t_1} \left(\frac{a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2}{2} - V\right) dt$$

is minimised, where a, b, c and V are prescribed functions of x, y and t. Write down the Euler equations for the problem.

If a, b, c and V are independent of t, show that

$$\frac{a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2}{2} + V$$

is constant during the motion.

If the end-point conditions are changed so that $y(t_1)$ is no longer prescribed to be y_1, what is the new boundary condition that must be applied to the Euler equations in order to find the minimising path?

Sturm-Liouville theorem, with application to eigenfunction expansions. Bessel functions and Legendre, Hermite and Laguerre polynomials as examples of systems of eigenfunctions. Sturm-Liouville equation derived from a variational principle.

13. Describe the part played in the theory of the Sturm-Liouville equation

$$Ly(x) + \lambda\rho(x)y(x) = 0 \tag{8}$$

and its non-homogeneous counterpart

$$Ly(x) + \lambda\rho(x)y(x) = f(x) \tag{9}$$

by the theory of Green's functions, the Fredholm Alternative Theorem, the Hilbert-Schmidt Expansion Theorem and the Calculus of Variations.

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